

ZERO COMMUTING MATRIX WITH PASCAL MATRIX

EUNMI CHOI*

ABSTRACT. In this work we study 0-commuting matrix $C(0)$ under relationships with the Pascal matrix $C(1)$. Like k th power $C(1)^k$ is an arithmetic matrix of $(kx + y)^n$ with $yx = xy$ ($k \geq 1$), we express $C(0)^k$ as an arithmetic matrix of certain polynomial with $yx = 0$.

1. Introduction

An arithmetic matrix of a polynomial $f(x)$ is a matrix consisting of coefficients of $f(x)$. The Pascal matrix is a famous arithmetic matrix of $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$, in which the commutativity $yx = xy$ is assumed tacitly. As a generalization, with two q -commuting variables x and y satisfying $yx = qxy$ ($q \in \mathbb{Z}$), the arithmetic matrix of $(x + y)^n$ is called a q -commuting matrix denoted by $C(q)$ [1]. The $C(q)$ is composed of q -binomials $\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{(1-q^i)(1-q^{i-1})\cdots(1-q^{i-j+1})}{(1-q)(1-q^2)\cdots(1-q^j)}$ [7]. When $q = \pm 1$, $C(1)$ is the Pascal matrix and $C(-1)$ is the Pauli Pascal matrix [4]. A block matrix form $C(-1) = \begin{bmatrix} i & & & \\ i & i & & \\ i & 2i & i & \\ i & 3i & 3i & i \\ & & \dots & \end{bmatrix}$ with a 2×2 matrix $i = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ looks very similar to $C(1)$ [5]. In particular if $q = 0$ then $C(0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$. The $n \times n$ matrix $C(0)$ was studied as an adjacency matrix of relation \leq on $\{1, 2, \dots, n\}$ and its inverse $C(0)^{-1} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \dots & 1 \end{bmatrix}$ is a sum of nilpotent matrix and identity matrix ([2], p.5)

Received May 12, 2020; Accepted August 20, 2020.

2010 Mathematics Subject Classification: 11B37, 11B39.

Key words and phrases: q -commuting matrix, Pauli Pascal matrix, arithmetic matrix.

In the work we simply denote $C(0)$ by Z and the Pascal matrix $C(1)$ by P , and study Z under relationships with P . Since P^k is an arithmetic matrix of $(kx + y)^n$ with $yx = xy$ for $k \geq 1$ [6], we develop interconnections of Z^k with P as well as an arithmetic matrix of certain polynomial.

For our notations, write $P = [e_{i,j}]$ and inverse $P^{-1} = [v_{i,j}]$, so $v_{i,j} = (-1)^{i-j}e_{i,j} = (-1)^{i-j}\binom{i}{j}$ for $i \geq j \geq 0$. As usual, M_n denotes a $n \times n$ matrix M .

2. Z^k ($k = 2, 3$) as arithmetic matrices

Note $Z = \begin{bmatrix} 1 \\ 11 \\ 111 \\ 1111 \end{bmatrix}$, $Z^2 = \begin{bmatrix} 1 \\ 21 \\ 321 \\ 4321 \end{bmatrix}$ and $Z^3 = \begin{bmatrix} 1 \\ 3 \ 1 \\ 6 \ 31 \\ 10631 \end{bmatrix}$. For each $k \geq 1$, write $Z^k = [y_0^{(k)} | y_1^{(k)} | y_2^{(k)} | \dots]$ where $y_j^{(k)}$ ($j \geq 0$) are j th column vectors.

LEMMA 2.1. Let $Y = [y_0^{(1)} | y_0^{(2)} | y_0^{(3)} | \dots] = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \dots \\ y_{2,1} & y_{2,2} & y_{2,3} & \dots \\ y_{3,1} & y_{3,2} & y_{3,3} & \dots \\ \dots & & & \end{bmatrix}$ be a matrix composed of 0th column vector $y_0^{(k)}$ of each Z^k ($k \geq 1$). Then $y_{i,j}$ ($i, j \geq 1$) satisfies $y_{i,j} = y_{i-1,j} + y_{i,j-1}$. In the k th column $y_0^{(k)}$, the i th entry is $y_{i,k} = e_{i+k-2,k-1}$.

Proof. Clearly $Y = \begin{bmatrix} 11 & 1 & 1 & 1 \\ 12 & 3 & 4 & 5 \\ 13 & 6 & 10 & 15 \\ 14 & 10 & 20 & 35 \end{bmatrix}$ is a symmetric type Pascal matrix satisfying $y_{i,j} = y_{i-1,j} + y_{i,j-1}$ [3]. From $y_0^{(3)} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 10 \\ \dots \end{bmatrix}$, $y_0^{(4)} = \begin{bmatrix} 1 \\ 4 \\ 10 \\ 20 \\ \dots \end{bmatrix}$ and $y_0^{(5)} = \begin{bmatrix} 1 \\ 5 \\ 15 \\ 35 \\ \dots \end{bmatrix}$, we notice $y_{i,3} = \frac{i(i+1)}{2} = \binom{i+1}{2}$, $y_{i,4} = \frac{i(i+1)(i+2)}{6} = \binom{i+2}{3}$ and $y_{i,5} = \binom{i+3}{4}$ in $y_0^{(k)}$ ($k = 3, 4, 5$). In $y_0^{(k)}$, assume $y_{i-1,k} = \binom{i+k-3}{k-1} = e_{i+k-3,k-1}$. Then we have $y_{i,k} = y_{i-1,k} + y_{i,k-1} = e_{i+k-3,k-1} + e_{i+k-3,k-2} = e_{i+k-2,k-1}$. □

Thus Z^k is determined by $y_0^{(k)} = \begin{bmatrix} e_{k-1,k-1} \\ e_{k,k-1} \\ e_{k+1,k-1} \\ \dots \end{bmatrix}$ by shifting one space down at each column. For Pascal matrix $P = C(1)$ of $(x+y)^n$ with $yx = xy$, the power P^k is an arithmetic matrix of $(kx+y)^n$. Indeed $P^2 = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix}$, $P^3 = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 9 & 6 & 1 & & \\ 27 & 27 & 9 & 1 & \\ & \dots & & & \end{bmatrix}$ yield expansions of $(2x+y)^n$ and $(3x+y)^n$. An analogy question is that how $Z^k = C(0)^k$ is related to polynomials $(kx+y)^n$ with $yx = 0$.

For $k \geq 1$, let $A^{(k)}$ be an arithmetic matrix of $(kx+y)^n$ with $yx = 0$.

THEOREM 2.2. *Let $\text{diag}[k^i]$ be a diagonal matrix having diagonal entries k^i ($i \geq 0$). Then $A^{(k)} = \text{diag}[k^i] Z \text{diag}[k^i]^{-1}$ and $A^{(k)} Z = Z A^{(k)}$.*

Proof. Observe $A^{(2)} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2^2 & 2 & 1 & \\ 2^3 & 2^2 & 2 & 1 \end{bmatrix}$ and $A^{(3)} = \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 3^2 & 3 & 1 & \\ 3^3 & 3^2 & 3 & 1 \end{bmatrix}$. Now for any k , assume $(kx+y)^n = k^n x^n + k^{n-1} x^{n-1} y + \dots + k x y^{n-1} + y^n$. Then $(kx+y)^{n+1} = (kx+y)^n (kx+y) = k^{n+1} x^{n+1} + k^n x^n y + \dots + k x y^n + y^{n+1}$, so $A^{(k)} = \begin{bmatrix} 1 & & & \\ k & 1 & & \\ k^2 & k & 1 & \\ k^3 & k^2 & k & 1 \end{bmatrix} = \text{diag}[k^i] Z \text{diag}[k^i]^{-1}$. Thus $Z A^{(k)} = A^{(k)} Z$ since

sum of entries of 0th column in $A_n^{(k)}$ equals that of n th row, and all entries of Z are 1. □

Let $J = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 1 & 0 & \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} 0 \\ \dots \\ 1 \end{bmatrix}$. Then $J^k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_k \end{bmatrix}$ for $k > 0$ by taking proper size matrices. Clearly J_n is a nilpotent matrix satisfying $J_n^k = 0$ for $k \geq n$, and we may consider $J^0 = I$.

THEOREM 2.3. *Let $\widehat{A}_k^{(2)} = A_k^{(2)} + J_k^{2T}$ and $\widehat{Z}_k^2 = -Z_k^2 + J_k^{2T}$. Then for any $n \geq 4$, $A_n^{(2)} = Z_n^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A}_{n-2}^{(2)} & J_{(n-2) \times 2}^{n-4} \end{array} \right]$ and $Z_n^2 = A_n^{(2)} \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline Z_{n-2}^2 & J_{(n-2) \times 2}^{n-4} \end{array} \right]$.*

Proof. The transpose matrix $J_k^{2T} = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & & & \dots & 0 \\ & & & & & \dots & 0 \\ 0 & & & & & & \dots & 0 \end{bmatrix}$ is upper triangular, so with lower triangular matrices $A_k^{(2)} = [a_{i,j}]$ and $Z_k^2 = [z_{i,j}]$, the $\widehat{A}_k^{(2)} =$

$[\hat{a}_{i,j}] = A_k^{(2)} + J_k^{2T}$ and $\widehat{Z}_k^2 = [\hat{z}_{i,j}] = -Z_k^2 + J_k^{2T}$ satisfy $\hat{a}_{i,j} = a_{i,j}$, $\hat{z}_{i,j} = -z_{i,j}$ if $i \geq j$, and $\hat{a}_{i,j} = \hat{z}_{i,j} = 1$ or 0 according to $i = j + 1$ or otherwise. Indeed,

$$\widehat{A}_k^{(2)} = A_k^{(2)} + J_k^{2T} = \begin{bmatrix} 1 & & & & \\ \frac{1}{2} & 1 & & & \\ \frac{2^2}{2} & 2 & 1 & & \\ \dots & & & & \\ 2^{k-1} & 2^{k-2} & \dots & 1 & \end{bmatrix} + \begin{bmatrix} 0010 \\ 0001 \\ \dots & 0 \\ \dots & 1 \\ 0 \dots 0 \\ 0 \dots 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & & \\ \frac{1}{2} & 1 & 0 & 1 & \\ \frac{2^2}{2} & 2 & 1 & 0 & 1 \\ \dots & & & & \\ 2^{k-3} & \dots & 2101 & & \\ 2^{k-2} & 2^{k-3} & \dots & 210 & \\ 2^{k-1} & 2^{k-2} & \dots & & 21 \end{bmatrix}.$$

When $n = 5$, simple multiplications of $A_5^{(2)}$ and Z_5 show that

$$Z_5^{-2} A_5^{(2)} = \begin{bmatrix} 1 \\ 01 \\ 101 \\ 2101 \\ 42101 \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A}_3^{(2)} & J_{3 \times 2} \end{array} \right] \text{ and } A_5^{(2)-1} Z_5^2 = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ -1 & 0 & 1 & & \\ -2 & -1 & 0 & 1 & \\ -3 & -2 & -1 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z}_3^2 & J_{3 \times 2} \end{array} \right],$$

because $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = A_3^{(2)} + J_3^{2T} = \widehat{A}_3^{(2)}$ and $\begin{bmatrix} -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \end{bmatrix} = -Z_3^2 + J_3^{2T} = \widehat{Z}_3^2$.

Now we write $Z_n^2 = \left[\begin{array}{c|c} Z_{n-1}^2 & 0 \\ \hline n, n-1, \dots, 2 & 1 \end{array} \right]$ and $A_n^{(2)} = \left[\begin{array}{c|c} A_{n-1}^{(2)} & 0 \\ \hline 2^{n-1}, \dots, 2^2, 2 & 1 \end{array} \right]$ in block matrix forms, and assume an induction hypothesis that

$$Z_n^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A}_{n-2}^{(2)} & J_{(n-2) \times 2}^{n-4} \end{array} \right] = A_n^{(2)} \text{ with } J_{(n-2) \times 2}^{n-4} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{n-4}.$$

Then Theorem 2 says

$$\begin{bmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{4}{3} & \frac{1}{3} & 1 & & \\ & 2 & 1 & & \\ n & n-1 & \dots & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ \dots & \dots & \dots & \dots \\ 2^{n-3} & 2^{n-4} & \dots & 2, 1, 0, 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{2}{3} & \frac{1}{2} & \frac{1}{2} & 1 & \\ & 2 & 1 & & \\ 2^{n-1} & 2^{n-2} & \dots & 2 & 1 \end{bmatrix}.$$

But since the $(n-1) \times (n+1)$ block matrix $\left[\begin{array}{c|c} \widehat{A}_{n-1}^{(2)} & J_{(n-1) \times 2}^{n-3} \end{array} \right]$ equals

$$\begin{bmatrix} 1 & 0 & 1 & \dots & 00 \\ 2 & 1 & 01 & \dots & 00 \\ \dots & \dots & \dots & \dots & 00 \\ 2^{n-4} & \dots & 2101 & 00 & \\ 2^{n-3} & 2^{n-4} & \dots & 21010 & \\ 2^{n-2} & 2^{n-3} & \dots & 2101 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & \dots & 00 \\ 2 & 1 & 01 & \dots & 00 \\ \dots & \dots & \dots & \dots & 00 \\ 2^{n-4} & \dots & 2101 & 00 & \\ 2^{n-3} & 2^{n-4} & \dots & 21010 & \\ 2^{n-2} & 2^{n-3} & \dots & 2101 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \widehat{A}_{n-2}^{(2)} & J_{(n-2) \times 2}^{n-4} \\ \hline 2^{n-2} \dots 2 : 1 & 0 \\ & 1 \end{array} \right],$$

the block matrix multiplications show

$$Z_{n+1}^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A}_{n-1}^{(2)} & J_{(n-1) \times 2}^{n-3} \end{array} \right] = \left[\begin{array}{c|c} Z_n^2 & 0 \\ \hline n+1, n, \dots, 2 & 1 \end{array} \right] \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A}_{n-2}^{(2)} & J_{(n-2) \times 2}^{n-4} \\ \hline 2^{n-2} \dots 2 : 1 & 0 \\ & 1 \end{array} \right] = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array} \right],$$

where $H = 0$, $S = 1$ and $G = Z_n^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A}_{n-2}^{(2)} & J_{(n-2) \times 2}^{n-4} \end{array} \right] = A_n^{(2)}$ by hypothesis.

And

$$\begin{aligned}
 R &= (n+1, n, \dots, 3, 2) \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 2^{n-4} & 2^{n-5} & \dots & 1 & 0 & 1 & 0 \\ 2^{n-3} & 2^{n-4} & \dots & 2 & 1 & 0 & 1 \end{bmatrix} + (2^{n-2}, \dots, 2^2, 2, 1, 0) \\
 &= \left(n+1 + \sum_{i=0}^{n-3} (n-1-i)2^i, n + \sum_{i=0}^{n-4} (n-2-i)2^i, n-1 + \sum_{i=0}^{n-5} (n-3-i)2^i, \right. \\
 &\quad \left. \dots, 4 + \sum_{i=0}^1 (2-i)2^i, 3, 2 \right) + (2^{n-2}, \dots, 2, 1, 0) \\
 &= \left(n+1 + \sum_{i=0}^{n-2} (n-1-i)2^i, n + \sum_{i=0}^{n-3} (n-2-i)2^i, n-1 + \sum_{i=0}^{n-4} (n-3-i)2^i, \right. \\
 &\quad \left. \dots, 4 + \sum_{i=0}^1 (2-i)2^i, 4, 2 \right).
 \end{aligned}$$

Thus the first few entries in R from right are $2, 2^2, 2^3, 2^4, \dots$. So as an induction hypothesis, we assume $n + \sum_{i=0}^{n-3} (n-2-i)2^i = 2^{n-1}$. Then

$$\begin{aligned}
 (n+1) + \sum_{i=0}^{n-2} (n-1-i)2^i &= (n+1) + (n-1) + \sum_{i=1}^{n-2} (n-1-i)2^i \\
 &= 2n + 2 \sum_{i=0}^{n-3} (n-2-i)2^i = 2n + 2(2^{n-1} - n) =
 \end{aligned}$$

2^n .

So $R = (2^n, 2^{n-1}, \dots, 2^3, 2^2, 2)$, thus $Z_{n+1}^2 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-1}^{(2)}} & J_{(n-1) \times 2}^{n-3} \end{array} \right] = A_{n+1}^{(2)}$.

And the second identity can be proved similarly. □

We now go on Z^3 and the arithmetic matrix $A^{(3)}$ of $(3x + y)^n$ with $yx = 0$.

THEOREM 2.4. Let $\widehat{A_k^{(3)}} = (3I_k - J_k)A_k^{(3)} + J_k^{2T}$ and $\widehat{Z_k^3} = -(3I_k - J_k)Z_k^3 + J_k^{2T}$. Then $A_n^{(3)} = Z_n^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2) \times 2}^{n-4} \end{array} \right]$ and $Z_n^3 = A_n^{(3)} \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z_{n-2}^{(3)}} & J_{(n-2) \times 2}^{n-4} \end{array} \right]$ for $n \geq 4$.

Proof. When $k = 5$, Theorem 2 shows $Z_5 A_5^{(3)} = A_5^{(3)} Z_5$ and

$$Z_5^{-3} A_5^{(3)} = \begin{bmatrix} 1 \\ 0 & 1 \\ 3 & 0 & 1 \\ 8 & 3 & 0 & 1 \\ 24 & 8 & 3 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X & J_{3 \times 2} \end{array} \right] \text{ with } X = \begin{bmatrix} 3 & 0 & 1 \\ 8 & 3 & 0 \\ 24 & 8 & 3 \end{bmatrix}.$$

Clearly $X = \begin{bmatrix} 3 & 00 \\ 8 & 30 \\ 24 & 83 \end{bmatrix} + \begin{bmatrix} 001 \\ 000 \\ 000 \end{bmatrix} = Y + J_3^{2T}$ with $Y = \begin{bmatrix} 3 & 00 \\ 8 & 30 \\ 24 & 83 \end{bmatrix}$. But since $YA_3^{(3)} = A_3^{(3)}Y$ and $A_3^{-(3)}Y = \begin{bmatrix} 3 \\ -1 & 3 \\ -13 \end{bmatrix} = 3I - J$, we have

$$X = Y + J_3^{2T} = A_3^{(3)}(3I - J) + J_3^{2T} = (3I - J)A_3^{(3)} + J_3^{2T} = \widehat{A_3^{(3)}},$$

so $Z_5^{-3}A_5^{(3)} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_3^{(3)}} & J_{3 \times 2}^{n-4} \end{array} \right]$. For some $n > 0$, we assume

$$A_n^{(3)} = Z_n^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2) \times 2}^{n-4} \end{array} \right] = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 6 & 3 & 1 & & \\ 10 & 6 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n+1}{2} & \binom{n}{2} & \cdots & 3 & 1 \end{bmatrix} \left[\begin{array}{c|c|c|c} 1 & & & \\ \hline 0 & 1 & & \\ \hline 3 & 0 & 1 & \\ \hline 8 & 3 & 0 & 1 \\ \hline 2^3 3^{n-4} & 2^3 3^{n-5} & \cdots & 2^3 3 0 1 \end{array} \right].$$

We note that

$$\begin{aligned} \widehat{A_n^{(3)}} &= (3I - J)A_n^{(3)} + J_n^{2T} \\ &= (3I - J) \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 3^2 & 3 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 3^{n-1} & 3^{n-2} & \cdots & 3 & 1 \end{bmatrix} + \begin{bmatrix} 0010 \cdots 0 \\ 0001 \cdots 0 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 8 & 3 & 0 & 1 \\ \cdots & \cdots & \cdots & 00 \\ 2^3 3^{n-3} & 2^3 3^{n-4} & \cdots & 3 0 1 0 \\ 2^3 3^{n-2} & 2^3 3^{n-3} & \cdots & 2^3 3 0 1 \end{bmatrix}. \end{aligned}$$

so the block matrix $\left[\begin{array}{c|c} \widehat{A_{n-1}^{(3)}} & J_{(n-1) \times 2}^{n-3} \end{array} \right]$ equals

$$\begin{aligned} \left[\begin{array}{c|c} 3 & 0 & 1 & 0 & 00 \\ 8 & 3 & 0 & 1 & 00 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2^3 3^{n-4} & 2^3 3^{n-5} & \cdots & 3, 0, 1, 0 & 10 \\ 2^3 3^{n-3} & 2^3 3^{n-4} & \cdots & 2^3, 3, 0, 1 & 01 \end{array} \middle| \begin{array}{c} 00 \\ 00 \\ \cdots \\ 10 \\ 01 \end{array} \right] &= \left[\begin{array}{c|c|c|c} 3 & 0 & 1 & 0 & 00 \\ 8 & 3 & 0 & 1 & 00 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2^3 3^{n-4} & 2^3 3^{n-5} & \cdots & 3, 0, 1, 0 & 10 \\ 2^3 3^{n-3} & 2^3 3^{n-4} & \cdots & 2^3, 3, 0, 1 & 01 \end{array} \middle| \begin{array}{c} 0 \\ 0 \\ \cdots \\ 10 \\ 01 \end{array} \right] \\ &= \left[\begin{array}{c|c} \widehat{A_{n-2}^{(3)}} & J_{(n-2) \times 2}^{n-4} \\ \hline 2^3 3^{n-3}, \cdots, 2^3, 3, 0 : 1, 0 & 1 \end{array} \right]. \end{aligned}$$

Thus the block matrix multiplications yield

$$\begin{aligned} Z_{n+1}^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-1}^{(3)}} & J_{(n-1) \times 2}^{n-3} \end{array} \right] &= \left[\begin{array}{c|c} Z_n^3 & 0 \\ \hline \binom{n+2}{2}, \binom{n+1}{2}, \cdots, 3 & 1 \end{array} \right] \left[\begin{array}{c|c} I_2 : 0 & : 0 \\ \hline \widehat{A_{n-2}^{(3)}} & : J_{(n-2) \times 2}^{n-4} \\ \hline 2^3 3^{n-3}, \cdots, 2^3, 3 : 1, 0 & 0 \\ & 1 \end{array} \right] \\ &= \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array} \right], \end{aligned}$$

where $G = Z_n^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2) \times 2}^{n-4} \end{array} \right] = A_n^{(3)}$, $H = 0_{n \times 1}$ and $S = 1$. And for R ,

$$R = \left(\binom{n+2}{2}, \binom{n+1}{2}, \cdots, 6, 3 \right) \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{A_{n-2}^{(3)}} & J_{(n-2) \times 2}^{n-4} \end{array} \right] + (2^3 3^{n-3}, \cdots, 2^3, 3, 0, 1, 0)$$

$$\begin{aligned}
&= \left(\binom{n+2}{2}, \binom{n+1}{2}, \dots, 6, 3 \right) \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 3 & 0 & 1 & & & \\ 8 & 3 & 0 & 1 & & \\ & \dots & & & & \\ 2^3 3^{n-4}, 2^3 3^{n-5}, \dots, 3, 0, 1 \end{bmatrix} + (2^3 3^{n-3}, \dots, 3, 0, 1, 0) \\
&= (\theta_1, \theta_2, \dots, \theta_n, \theta_{n+1}) + (2^3 3^{n-3}, \dots, 2^3, 3, 0, 1, 0),
\end{aligned}$$

where

$$\begin{aligned}
\theta_1 &= \binom{n+2}{2} + 3 \binom{n}{2} + 2^3 \binom{n-1}{2} + 2^3 3 \binom{n-2}{2} + \dots + 2^3 3^{n-4} \binom{3}{2}, \\
\theta_2 &= \binom{n+1}{2} + 3 \binom{n-1}{2} + 2^3 \binom{n-2}{2} + 2^3 3 \binom{n-3}{2} + \dots + 2^3 3^{n-5} \binom{3}{2}, \dots, \\
\theta_{n-2} &= \frac{6 \binom{5}{2}}{2} + 6 \cdot 3 + 3 \cdot 2^3, \quad \theta_{n-1} = \frac{5 \binom{4}{2}}{2} + 3, \quad \theta_n = 6, \quad \theta_{n+1} = 3.
\end{aligned}$$

So the last few entries in R are $\theta_{n+1} = 3$, $\theta_n + 3 = 3^2$, $\theta_{n-1} + 2^3 = 3^3$ and $\theta_{n-2} + 2^3 \cdot 3 = 3^4$. We now assume $3^{n-1} = \theta_2 + 2^3 3^{n-4}$. Then

$$3^{n-1} = \left(\binom{n+1}{2} + 3 \binom{n-1}{2} + 2^3 \binom{n-2}{2} + 2^3 3 \binom{n-3}{2} + \dots + 2^3 3^{n-5} \binom{3}{2} \right) + 2^3 3^{n-4}.$$

Hence

$$\begin{aligned}
&\theta_1 + 2^3 3^{n-3} \\
&= \left(\binom{n+2}{2} + 3 \binom{n}{2} + 2^3 \binom{n-1}{2} + 2^3 3 \binom{n-2}{2} + \dots + 2^3 3^{n-4} \binom{3}{2} \right) + 2^3 3^{n-3} \\
&= \binom{n+2}{2} + 3 \binom{n}{2} + 2^3 \binom{n-1}{2} + 3 \cdot 2^3 \left(\binom{n-2}{2} + 3 \binom{n-3}{2} + \dots + 3^{n-5} \binom{3}{2} + 3^{n-4} \right) \\
&= \binom{n+2}{2} + 3 \binom{n}{2} + 2^3 \binom{n-1}{2} + 3 \left(3^{n-1} - \binom{n+1}{2} - 3 \binom{n-1}{2} \right) \\
&= 3^n + \left(\binom{n+2}{2} + 3 \binom{n}{2} + 2^3 \binom{n-1}{2} - 3 \binom{n+1}{2} - 3^2 \binom{n-1}{2} \right) = 3^n,
\end{aligned}$$

by induction hypothesis. Thus $R = (3^n, 3^{n-1}, \dots, 3^2, 3)$, so we have

$$Z_{n+1}^3 \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline A_{n-1}^{(3)} & J_{(n-1) \times 2}^{n-3} \end{array} \right] = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array} \right] = \left[\begin{array}{c|c} A_n^{(3)} & 0 \\ \hline 3^n, 3^{n-1}, \dots, 3 & 1 \end{array} \right] = A_{n+1}^{(3)}.$$

Similarly for the second identity, we have

$$A^{(3)-1} Z^3 = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ -3 & 0 & 1 & & \\ -8 & -3 & 0 & 1 & \\ -15 & -8 & -3 & 0 & 1 \\ \dots & & & & \end{bmatrix} = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X & J \end{array} \right] \text{ with } X = \begin{bmatrix} -3 & 0 & 1 \\ -8 & -3 & 0 \\ -15 & -8 & -3 \\ \dots & & \end{bmatrix}.$$

$$\text{But since } X = \begin{bmatrix} -3 & 0 & 0 \\ -8 & -3 & 0 \\ -15 & -8 & -3 \\ \dots & & \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dots & & \end{bmatrix} = -Y + J^{2T} \text{ with } Y = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 3 & 0 \\ 15 & 8 & 3 \\ \dots & & \end{bmatrix},$$

$$\text{we have } YZ^{-3} = \begin{bmatrix} 3 & & & \\ -1 & 3 & & \\ & -13 & & \\ \dots & & & \end{bmatrix} = 3I - J. \text{ So } Y = Z^3(3I - J) = (3I - J)Z^3$$

$$\text{and } X = -(3I - J)Z^3 + J^{2T} = \widehat{Z}^3. \text{ Thus } Z_n^3 = A_n^{(3)} \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z}_{n-2}^3 & J_{(n-2) \times 2}^{n-4} \end{array} \right]. \quad \square$$

Connections of $A^{(k)}$ and Z^k ($k = 2, 3$) are further explained by J .

THEOREM 2.5. (1) $Z_n^{-2}A_n^{(2)} = I_n + J_n^2A_n^{(2)}$ and $A_n^{(2)^{-1}}Z_n^2 = I_n - J_n^2Z_n^2$.

(2) $Z_n^{-3}A_n^{(3)} = I_n + J_n^2(3I_n - J_n)A_n^{(3)}$ and $A_n^{(3)^{-1}}Z_n^3 = I_n - J_n^2(3I_n - J_n)Z_n^3$.

(3) $A_n^{(2)^{-1}} = Z_n^{-2} - J_n^2$ and $A_n^{(3)^{-1}} = Z_n^{-3} - J_n^2(3I_n - J_n)$.

Proof. With $\widehat{A_k^{(2)}} = A_k^{(2)} + J_k^{2T}$, Theorem 3 shows

$$\begin{aligned} Z_n^{-2}A_n^{(2)} &= \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline A_{n-2}^{(2)} & J_{(n-2)\times 2}^{n-4} \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \hline 1 & 0 & 1 & \\ 2 & 1 & 0 & 1 \\ \dots & & & 10 \\ 2^{n-3} & 2^{n-4} & \dots & 01 \end{array} \right] = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline A_{n-2}^{(2)} \end{array} \right] \\ &= I_n + J_n^2A_n^{(2)}. \end{aligned}$$

Similarly with $\widehat{Z_k^2} = -Z_k^2 + J_k^{2T}$, we also have

$$\begin{aligned} A_n^{(2)^{-1}}Z_n^2 &= \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z_{n-2}^2} & J_{(n-2)\times 2}^{n-4} \end{array} \right] = \left[\begin{array}{ccc|c} 1 & & 00 & \\ 0 & & 10 & \\ \hline -1 & & 01 & \\ -2 & & -10 & 1 \\ \dots & & \dots & 10 \\ -e_{n-2,1} & -e_{n-3,1} & \dots & 01 \end{array} \right] = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline -Z_{n-2}^2 \end{array} \right] \\ &= I_n - J_n^2Z_n^2. \end{aligned}$$

On the other hand, with $\widehat{A_k^{(3)}} = (3I_k - J_k)A_k^{(3)} + J_k^{2T}$, Theorem 4 shows

$$\begin{aligned} Z_n^{-3}A_n^{(3)} &= \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline A_{n-2}^{(3)} & J_{(n-2)\times 2}^{n-4} \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \hline 3 & 0 & 1 & \\ 8 & 3 & 0 & 1 \\ 24 & 8 & 30 & 1 \\ \dots & & & 10 \\ 2^3 & 3^{n-4} & 2^3 & 2^{n-5} \dots \dots 01 \end{array} \right] = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline (3I - J)A_{n-2}^{(3)} \end{array} \right] \\ &= I_n + J_n^2(3I_n - J_n)A_n^{(3)}. \end{aligned}$$

Moreover $A_n^{(3)^{-1}}Z_n^3 = \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline \widehat{Z_{n-2}^3} & J_{(n-2)\times 2}^{n-4} \end{array} \right] = I_n - J_n^2(3I - J)Z_n^3$. And also

$Z_n^{-2} = A_n^{(2)^{-1}} + J_n^2$ and $Z_n^{-3} = A_n^{(3)^{-1}} + J_n^2(3I_n - J_n)$ by (1) and (2). \square

In fact, $Z^{-2} - J^2 = \begin{bmatrix} -\frac{1}{2} & 1 & \\ 1 & -2 & 1 \\ & 1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 0 & 100 & \\ & 100 & \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 & \\ -2 & -2 & 1 \\ & -2 & -2 \end{bmatrix} = A^{(2)^{-1}}$.

3. Z^k as an arithmetic matrix

Theorem 5 says $Z_n^{-2}A_n^{(2)} = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline A_{n-1}^{(2)} \end{array} \right] = I_n + J_n^2A_n^{(2)}$,

$$Z_n^{-3}A_n^{(3)} = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline (3I_{n-2} - J_{n-2})A_{n-2}^{(3)} \end{array} \right] = I_n + J_n^2(3I_n - J_n)A_n^{(3)},$$

$$A_n^{(2)-1}Z_n^2 = I_n - J_n^2Z_n^2 \text{ and } A_n^{(3)-1}Z_n^3 = I_n - J_n^2(3I_n - J_n)Z_n^3.$$

In order to generalize the identities, first consider $k = 4, 5, 6$. Then we have

$$Z_n^{-k}A_n^{(k)} = I_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline S^{(k)} \end{array} \right] = I_n + J_n^2S^{(k)} \tag{1}$$

$$\text{with } S^{(4)} = \left[\begin{array}{ccc} 6 & & \\ 20 & 6 & \\ 81 & 206 & \\ 324 & 81 & 206 \\ \dots & & \end{array} \right], S^{(5)} = \left[\begin{array}{ccc} 10 & & \\ 40 & 10 & \\ 205 & 40 & 10 \\ 1024 & 205 & 40 & 10 \\ \dots & & & \end{array} \right] \text{ and } S^{(6)} = \left[\begin{array}{ccc} 15 & & \\ 70 & 15 & \\ 435 & 70 & 15 \\ 2604 & 435 & 70 & 15 \\ \dots & & & \end{array} \right].$$

Also

$$A_n^{(k)-1}Z_n^k = I_n - \left[\begin{array}{c} 0 \\ 0 \\ \hline T^{(k)} \end{array} \right] = I_n - J_n^2T^{(k)} \tag{2}$$

$$\text{with } T^{(4)} = \left[\begin{array}{ccc} 6 & & \\ 20 & 6 & \\ 45 & 206 & \\ 84 & 45 & 206 \\ \dots & & \end{array} \right], T^{(5)} = \left[\begin{array}{ccc} 10 & & \\ 40 & 10 & \\ 105 & 40 & 10 \\ 224 & 105 & 40 & 10 \\ \dots & & & \end{array} \right] \text{ and } T^{(6)} = \left[\begin{array}{ccc} 15 & & \\ 70 & 15 & \\ 210 & 70 & 15 \\ 504 & 210 & 70 & 15 \\ \dots & & & \end{array} \right].$$

Now by means of the k th row $(v_{k,0}, \dots, v_{k,k})$ of P^{-1} , we define a matrix

$$X^{(k)} = \sum_{j=0}^{k-2} v_{k,j}J^{k-j-2}. \tag{3}$$

THEOREM 3.1. For $2 \leq k \leq 6$, $X_n^{(k)}A_n^{(k)} = X_n^{(k)}A_n^{(k)}$ and $X_n^{(k)}Z_n^k = Z_n^kX_n^{(k)}$. Moreover $Z_n^{-k}A_n^{(k)} = I_n + J_n^2X_n^{(k)}A_n^{(k)}$ and $A_n^{(k)-1}Z_n^k = I_n - J_n^2X_n^{(k)}Z_n^k$.

Proof. Clearly $X^{(2)} = I$ and $X^{(3)} = 3I - J$, so Theorem 5 implies

$$I_n + J_n^2X_n^{(2)}A_n^{(2)} = I_n + J_n^2A_n^{(2)} = Z_n^{-2}A_n^{(2)}$$

$$I_n + J_n^2X_n^{(3)}A_n^{(3)} = I_n + J_n^2(3I_n - J_n)A_n^{(3)} = Z_n^{-3}A_n^{(3)}$$

$$I_n - J_n^2X_n^{(2)}A_n^{(2)} = A_n^{(2)-1}Z_n^2 \text{ and } I_n + J_n^2X_n^{(3)}A_n^{(3)} = A_n^{(3)-1}Z_n^3.$$

Now from $Z_n^{-k}A_n^{(k)} = I_n + J_n^2S^{(k)}$ in (1), we have

$$A^{(4)-1}S^{(4)} = \left[\begin{array}{ccc} 6 & & \\ -4 & 6 & \\ 1 & -4 & 6 \\ & 1 & -4 & 6 \\ \dots & & & \end{array} \right] = 6I - 4J + J^2 = S^{(4)}A^{(4)-1}$$

$$A^{(5)-1}S^{(5)} = \left[\begin{array}{ccc} 10 & & \\ -10 & 10 & \\ 5 & -10 & 10 \\ -1 & 5 & -10 & 10 \\ & -1 & 5 & -10 & 10 \\ \dots & & & & \end{array} \right] = 10I - 10J + 5J^2 - J^3 = S^{(5)}A^{(5)-1}$$

$$A^{(6)^{-1}}S^{(6)} = \begin{bmatrix} 15 & & & \\ -20 & 15 & & \\ 15 & -20 & 15 & \\ -6 & 15 & -20 & 15 \\ \dots & \dots & \dots & \dots \end{bmatrix} = 15I - 20J + 15J^2 - 6J^3 + J^4.$$

Since each coefficients in expressions of $A^{(k)^{-1}}S^{(k)}$ correspond to elements in k th row of P^{-1} , we have $A^{(k)^{-1}}S^{(k)} = S^{(k)}A^{(k)^{-1}} = X^{(k)}$, so (1) gives

$$Z_n^{-k}A_n^{(k)} = I_n + J_n^2S^{(k)} = I_n + J_n^2X_n^{(k)}A_n^{(k)}.$$

Similarly each $T^{(k)}$ in (2) satisfies $Z^{-k}T^{(k)} = T^{(k)}Z^{-k} = A^{(k)^{-1}}S^{(k)} = X^{(k)}$. So we have $A_n^{-(k)}Z_n^k = I_n - J_n^2T^{(k)} = I_n - J_n^2X_n^{(k)}Z_n^k$. \square

THEOREM 3.2. Let $\mu = \sum_{j=0}^{k-2} k^j v_{k,j}$ for $k \geq 2$. Then $X^{(k)}A^{(k)}$ is equal

to

$$\begin{bmatrix} v_{k,k-2} \\ v_{k,k-3} + kv_{k,k-2}, v_{k,k-2} \\ \dots \\ \sum_{j=1}^{k-2} k^{j-1}v_{k,j}, \quad \sum_{j=2}^{k-2} k^{j-2}v_{k,j}, \quad \sum_{j=3}^{k-2} k^{j-3}v_{k,j}, \dots \\ \mu \quad \sum_{j=1}^{k-2} k^{j-2}v_{k,j}, \quad \sum_{j=2}^{k-2} k^{j-2}v_{k,j}, \dots \\ k\mu \quad \mu \quad \sum_{j=1}^{k-2} k^{j-1}v_{k,j}, \dots \\ k^2\mu \quad k\mu \quad \mu \quad \dots \\ \dots \\ k^{n-k+1}\mu \quad k^{n-k}\mu \quad \dots, \mu, \dots, v_{k,k-3} + kv_{k,k-2}, v_{k,k-2} \end{bmatrix}$$

Proof. We note that the first few $X^{(k)}$ in (3) are

$$X^{(4)} = v_{4,0}J^2 + v_{4,1}J + v_{4,2}I = \begin{bmatrix} 0 \\ 1 \\ 1 \\ \dots \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ \dots \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 1 \\ 1 \\ \dots \end{bmatrix} = \begin{bmatrix} -4 \\ 1 & -4 & 6 \\ \dots & \dots & 6 \end{bmatrix}$$

$$X^{(5)} = v_{5,0}J^3 + v_{5,1}J^2 + v_{5,2}J + v_{5,3}I = \begin{bmatrix} -10 & 10 \\ 5 & -10 & 10 \\ -1 & 5 & -10 & 10 \\ \dots & \dots & \dots & \dots \end{bmatrix} \text{ and also,}$$

$$X^{(6)} = \begin{bmatrix} -15 & & & \\ -20 & 15 & & \\ 15 & -20 & 15 & \\ -6 & 15 & -20 & 15 \\ 1 & -6 & 15 & -20 & 15 \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \text{ So } X^{(k)} = \begin{bmatrix} v_{k,k-2} & & & \\ v_{k,k-3} & v_{k,k-2} & & \\ \dots & \dots & \dots & \dots \\ v_{k,1} & v_{k,2} & \dots & v_{k,k-2} \\ v_{k,0} & v_{k,1} & \dots & \dots \\ & v_{k,0} & v_{k,1} & v_{k,2} & \dots & v_{k,k-2} \\ & & v_{k,0} & v_{k,1} & \dots & v_{k,k-3} \\ & & & \dots & \dots & \dots \end{bmatrix}$$

for any k . Then since $A^{(3)} = \begin{bmatrix} 1 \\ 3 & 1 \\ 3^2 & 3 & 1 \\ 3^3 & 3^2 & 3 & 1 \\ \dots & \dots & \dots & \dots \end{bmatrix}$ and $\mu = v_{3,0} + 3v_{3,1}$, we have

$$X^{(3)}A_n^{(3)} = \begin{bmatrix} v_{3,1} & & & & & & & & \\ v_{3,0} + 3v_{3,1} & v_{3,1} & & & & & & & \\ 3v_{3,0} + 3^2v_{3,1} & v_{3,0} + 3v_{3,1} & v_{3,1} & & & & & & \\ 3^2v_{3,0} + 3v_{3,1} \cdots & & & v_{3,1} & & & & & \\ \cdots & & & & & & & & \end{bmatrix} = \begin{bmatrix} v_{3,1} & & & & & & & & \\ \mu & v_{3,1} & & & & & & & \\ 3\mu & \mu & v_{3,1} & & & & & & \\ 3^2\mu & 3\mu & \mu & v_{3,1} & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 3^{n-2}\mu, 3^{n-3}\mu, \cdots & \mu, & v_{3,1} & & & & & & \end{bmatrix}.$$

Similarly $A^{(4)}$ and $\mu = \sum_{j=0}^2 4^j v_{4,j}$ imply

$$X^{(4)}A^{(4)} = \begin{bmatrix} v_{4,2} & & & & & & & & & & \\ v_{4,1} + 4v_{4,2}, v_{4,2} & & & & & & & & & & \\ \mu & v_{4,1} + 4v_{4,2}, v_{4,2} & & & & & & & & & \\ 4\mu & \mu & v_{4,1} + 4v_{4,2}, v_{4,2} & & & & & & & & \\ 4^2\mu & 4\mu & \mu & v_{4,1} + 4v_{4,2}, v_{4,2} & & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 4^{n-3}\mu & 4^{n-4}\mu & 4^{n-5}\mu & \cdots & \cdots & \cdots & \cdots & \cdots & \mu & v_{4,1} + 4v_{4,2}, v_{4,2} & \end{bmatrix}.$$

Thus for any $k > 1$, it follows that

$$\begin{aligned} X^{(k)}A^{(k)} &= (v_{k,0}J^{k-2} + v_{k,1}J^{k-3} + \cdots + v_{k,k-3}J + v_{k,k-2}I)A^{(k)} \\ &= v_{k,0} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{k-2} \left[\begin{array}{c} \\ \\ \\ \end{array} \right]_{A^{(k)}} + v_{k,1} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{k-3} \left[\begin{array}{c} \\ \\ \\ \end{array} \right]_{A^{(k)}} + \cdots + v_{k,k-3} \begin{bmatrix} 0 \\ A^{(k)} \end{bmatrix} + v_{k,k-2}A^{(k)}. \end{aligned}$$

So with $\mu = \sum_{j=0}^{k-2} k^j v_{k,j}$, $X^{(k)}A^{(k)}$ holds the required form in Theorem 7. □

We are ready to generalize Theorem 6 to Z^k and $A^{(k)}$ for all $k \geq 2$.

THEOREM 3.3. $Z^{-k}A^{(k)} = I + J^2X^{(k)}A^{(k)}$ and $A^{(k)-1}Z^k = I - J^2X^{(k)}Z^k$.

Proof. It is Theorem 6 if $2 \leq k \leq 6$. For any k , consider $A_n^{(k)}$ and

$$X_n^{(k)} = v_{k,0}J_n^{k-2} + \cdots + v_{k,k-4}J_n^2 + v_{k,k-3}J_n + v_{k,k-2}I_n.$$

Let $n = 3$. Since the nilpotent matrix $J_n^t = 0$ for $t \geq n$, we have

$$X_3^{(k)} = v_{k,k-4}J_n^2 + v_{k,k-3}J_n + v_{k,k-2}I = \begin{bmatrix} e_{k,k-2} & 0 & 0 \\ -e_{k,k-3} & e_{k,k-2} & 1 \\ e_{k,k-4} & -e_{k,k-3} & e_{k,k-2} \end{bmatrix}.$$

So with $A_3^{(k)} = \begin{bmatrix} 1 & & \\ k & 1 & \\ k^2 & k & 1 \end{bmatrix}$ and $Z_3^k = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ e_{k+1,k-1} & k & 1 \end{bmatrix}$ in Lemma 1, we

have

$$I_3 + J_3^2X_3^{(k)}A_3^{(k)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_{k,k-2} & 0 & 1 \end{bmatrix} \text{ and } Z_3^k(I_3 + J_3^2X_3^{(k)}A_3^{(k)}) = \begin{bmatrix} 1 & & \\ k & 1 & \\ k^2 & k & 1 \end{bmatrix} =$$

$A_3^{(k)}$.

Now for some n , as an induction hypothesis, we assume

$$A_n^{(k)} = Z_n^k(I_n + J_n^2X_n^{(k)}A_n^{(k)}) = Z_n^k \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X_{n-2}^{(k)}A_{n-2}^{(k)} + J_{n-2}^2 & J_{(n-2) \times 2}^{n-4} \end{array} \right].$$

By Lemma 1 we note that

$$Z_n^k = \left[\begin{array}{cc|c} 1 & & \\ e_{k,k-1} & 1 & \\ e_{k+1,k-1} & e_{k,k-1} & 1 \\ & \dots & \\ e_{k+(n-2),k-1} & \dots & e_{k,k-1} & 1 \end{array} \right] = \left[\begin{array}{c|c} Z_{n-1}^k & 0 \\ \hline e_{k+(n-2),k-1}, \dots, e_{k,k-1} & 1 \end{array} \right] \quad (4)$$

Then

$$\begin{aligned} Z_{n+1}^k(I_{n+1} + J_{n+1}^2 X_{n+1}^{(k)} A_{n+1}^{(k)}) &= Z_{n+1}^k \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X_{n-1}^{(k)} A_{n-1}^{(k)} + J_{n-1}^2 & J_{(n-1) \times 2}^T \end{array} \right] \\ &= \left[\begin{array}{c|c} Z_n^k & 0 \\ \hline Z' & 1 \end{array} \right] \left[\begin{array}{c|c} I_2 & : 0 \\ \hline X_{n-2}^{(k)} A_{n-2}^{(k)} + J_{n-2}^2 & J_{(n-2) \times 2}^T \end{array} \right] \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \end{aligned} \quad (5)$$

where bottom rows Z' and X' are from (4) and Theorem 7 that

$$\begin{aligned} Z' &= (e_{k+(n-1),k-1}, e_{k+(n-2),k-1}, \dots, e_{k+2,k-1}, e_{k+1,k-1}, e_{k,k-1}), \\ X' &= (k^{n-k}\mu, k^{n-k-1}\mu, \dots, k\mu, \mu, \sum_{j=1}^{k-2} k^{j-1}v_{k,j}, \sum_{j=2}^{k-2} k^{j-2}v_{k,j}, \\ &\dots, \sum_{j=k-3}^{k-2} k^{j-(k-3)}v_{k,j}, v_{k,k-2}, 0). \end{aligned} \quad (6)$$

Therefore the block matrix multiplications in (5) yield

$$Z_{n+1}^k(I_{n+1} + J_{n+1}^2 X_{n+1}^{(k)} A_{n+1}^{(k)}) = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array} \right]$$

with $H = 0$, $S = 1$ and $G = Z_n^k \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X_{n-2}^{(k)} A_{n-2}^{(k)} + J_{n-2}^2 & J_{(n-2) \times 2}^T \end{array} \right] = A_n^{(k)}$ by induction hypothesis. Now we claim $R = (k^n, k^{n-1}, \dots, k)$. If so, we get

$$Z_{n+1}^k(I_{n+1} + J_{n+1}^2 X_{n+1}^{(k)} A_{n+1}^{(k)}) = \left[\begin{array}{c|c} G & H \\ \hline R & S \end{array} \right] = \left[\begin{array}{c|c} A_n^{(k)} & 0 \\ \hline k^n, k^{n-1}, \dots, k & 1 \end{array} \right] = A_{n+1}^{(k)},$$

as required. Now write $R = Z' \left[\begin{array}{c|c} I_2 : 0 & 0 \\ \hline X_{n-2}^{(k)} A_{n-2}^{(k)} + J_{n-2}^2 & J_{(n-2) \times 2}^T \end{array} \right] + X'$ by

$$R = (\theta_1, \theta_2, \dots, \theta_n) + X' = (\chi_1, \chi_2, \dots, \chi_n), \quad (7)$$

with some $\theta_i, \chi_i \in \mathbb{Z}$. Due to Theorem 7, the multiplication

$$= B + (e_{n-2,k-1}\mu + e_{n-3,k-1}k\mu + \cdots + e_{k,k-1}k^{n-k-2}\mu + k^{n-k-1}\mu),$$

where $B = e_{k+(n-2),k-1} + e_{k+(n-4),k-1}v_{k,k-2} + e_{k+(n-5),k-1}(v_{k,k-3} + kv_{k,k-2}) + \cdots + e_{n,k-1} \sum_{j=2}^{k-2} k^{j-2}v_{k,j} + e_{n-1,k-1} \sum_{j=1}^{k-2} k^{j-1}v_{k,j}.$

Therefore

$$\chi_1 = A + k(e_{n-2,k-1}\mu + e_{n-3,k-1}k\mu + \cdots + e_{k,k-1}k^{n-k-2}\mu + k^{n-k-1}\mu) = A + k(k^{n-1} - B) = k^n + (A - kB).$$

But we have

$$\begin{aligned} & A - kB \\ &= (e_{k+(n-1),k-1} + e_{k+(n-3),k-1}v_{k,k-2} + e_{k+(n-4),k-1}(v_{k,k-3} + kv_{k,k-2}) \\ & \quad + \cdots + e_{n,k-1} \sum_{j=1}^{k-2} k^{j-1}v_{k,j} + e_{n-1,k-1}\mu) \\ & \quad - k(e_{k+(n-2),k-1} + e_{k+(n-4),k-1}v_{k,k-2} + e_{k+(n-5),k-1}(v_{k,k-3} + kv_{k,k-2}) \\ & \quad + \cdots + e_{n,k-1} \sum_{j=2}^{k-2} k^{j-2}v_{k,j} + e_{n-1,k-1} \sum_{j=1}^{k-2} k^{j-1}v_{k,j}) \\ &= e_{k+(n-1),k-1} - e_{k+(n-2),k-1}k + e_{k+(n-3),k-1}v_{k,k-2} \\ & \quad + e_{k+(n-4),k-1}((v_{k,k-3} + kv_{k,k-2}) - kv_{k,k-2}) + \cdots \\ & \quad + e_{n,k-1}(\sum_{j=1}^{k-2} k^{j-1}v_{k,j} - \sum_{j=2}^{k-2} k^{j-1}v_{k,j}) + e_{n-1,k-1}(\mu - \sum_{j=1}^{k-2} k^jv_{k,j}) \\ &= e_{k+(n-1),k-1} - e_{k+(n-2),k-1}k + e_{k+(n-3),k-1}v_{k,k-2} \\ & \quad + e_{k+(n-4),k-1}v_{k,k-3} + \cdots + e_{n,k-1}v_{k,1} + e_{n-1,k-1}v_{k,0} \\ &= \sum_{i=0}^k v_{k,k-i}e_{k+(n-i-1),k-1}. \end{aligned}$$

It means $A - kB$ is a multiplication of k th row in P^{-1} and $(k - 1)$ th column in P , so $A - kB = 0$ and we have $\chi_1 = k^n$, as required. \square

Therefore it follows that $Z^{-k} - A^{(k)^{-1}} = J^2 X^{(k)} = \begin{bmatrix} 0 \\ 0 \\ X^{(k)} \end{bmatrix}.$

References

[1] H. Cohn, *Projective geometry over F_1 and the gaussian binomial coefficients*, Amer. Math. Monthly, **111** (2004), 487 - 495.
 [2] Hans Cuypers, *Discrete Mathematics*, <https://www.coursehero.com/file/14299872/notes/> (2007), 1-74
 [3] A. Edelman and G. Strang, *Pascal matrices*, Amer. Math. Monthly, **111** (2004), 361 - 385.
 [4] M.E. Horn, *Pascal Pyramids, Pascal Hyper-Pyramids and a Bilateral Multinomial Theorem*, arXiv:math/0311035v1 (math.GM), **4** (2003).

- [5] M.E. Horn, *Pauli Pascal Pyramids, Pauli Fibonacci Numbers, and Pauli Jacobsthal Numbers*, arXiv:0711.4030v1 (math.GM), **26** (2007).
- [6] J. Jo, Y. Oh, and E. Choi, *Arithmetic matrix of quadratic polynomial with negative exponent by Pascal matrix*, J. of Algebra and Applied Mathematics, **17** (2019), 67-90.
- [7] T.H. Koornwinder, *Special functions and q -commuting variables*, in: Special Functions, q -Series and Related Topics, M.E.H. Ismail, D.R. Masson, M. Rahman (eds.), Fields Institute Communications **14**, AMS (1997), 131 -166.

*

Department of Mathematics
Hannam University
Daejeon, Republic of Korea
E-mail: emc@hnu.kr